UARK GSC What is...? #1: Tropical Geometry

Abstract

Ever since you were in the womb, you've known that addition and multiplication form primary building blocks for so much of mathematics: we can discuss arithmetic, polynomials, curves, vector spaces, rings, etc. To deviate from this must seem like pure blasphemy. But in *tropical geometry*, we do exactly this, and remarkably enough, don't throw the baby out with the bathwater. In this talk, we'll see a beginner-friendly exposition into what happens when you replace addition $x + y$ with $\min\{x, y\}$ and multiplication $x \cdot y$ with $x + y$ (and also maybe why the heck you'd wanna do such a thing). Based off of notes by Speyer and Strumfels.

Part One: Definitions

The ring $\mathbf{R}$ is known to even middle school or elementary students, though they might not use that language. We all know that we can add and multiply real numbers, subject to axioms like associativity, commutativity, distribution, etc. We disrupt this by considering a new almost ring structure on $\mathbf{R}$, where we have **tropical addition** $\oplus$ and **tropical multiplication** $\otimes$

$$\begin{aligned} x \oplus y &\coloneqq \min\{x, y\}, \text{ and} \\ x \otimes y &\coloneqq x + y. \end{aligned}$$

For example, we now have $3 \oplus 5 = \min\{3, 5\} = 3$ and $3 \otimes 5 = 3 + 5 = 8$.

We say that this introduces an almost ring structure because some of the ring axioms still hold. Taking minima is associative and commutative, as of course is addition, so tropical addition and tropical multiplication are associative and commutative. Also, we get a distribution law:

$$x \otimes (y \oplus z) = x + \min\{y, z\} = \min\{x + y, x + z\} = (x \otimes y) \oplus (x \otimes z).$$

Adopting the natural convention that our order of operations should do tropical multiplication before tropical addition, the parentheses on the right side are unnecessary.

We of course have a tropical multiplicative identity: it is 0. But in order to have a tropical additive identity, we need a number e such that $x \oplus e = \min\{x, e\} = x$ for all $x \in \mathbf{R}$. That is, we need a number e that is bigger than any real number. Thus ∞ is our tropical additive identity; throw it into our set.

But we don't get a complete ring structure, because we don't have tropical additive inverses; we can't tropically subtract. Finding "5 tropical-minus 3" amounts to solving for x:

$$\begin{aligned} 3 \oplus x &= 5 \\ \min\{3, x\} &= 5 \end{aligned}$$

which has no solution. Thus we have a tropical *semiring* $(\mathbf{R} \cup \{\infty\}, \oplus, \otimes)$.

Part Two: Arithmetic

Some common arithmetic constructions behave very differently in tropical geometry. For instance, if we want to build a tropical version of Pascal's triangle, where the nth row represents the binomial cofficients $\binom{n}{k}$, recall that we fill Pascal's triangle by placing 1s at the start and end of each row, and filling in terms by adding the nearest two vertical neighbors.

In tropical Pascal, we seed with 0s at the start and end of each row (the tropical multiplicative identity), and tropically add the nearest vertical neighbors. We end up with

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \\ \vdots & & & & & & & & \vdots \end{array}$$

since $0 \oplus 0 = \min\{0, 0\} = 0$. This also tells us tropical binomial coefficients: keeping in mind that exponentiation and concatenation are with respect to tropical multiplication, expanding out $(x \oplus y)^n$ will give us the expression

$$0x^n \oplus 0x^{n-1}y \oplus \cdots \oplus 0xy^{n-1} \oplus 0y^n.$$

Writing $0 \otimes$ anything is superfluous because it's $0 +$, so we get

$$x^n \oplus x^{n-1}y \oplus \cdots \oplus xy^{n-1} \oplus y^n.$$

We actually get the Freshman's dream in tropical geometry as well: $(x \oplus y)^n = x^n \oplus y^n$. So there's no need to bother with binomial coefficients. You can check this explicitly:

$$(x \oplus y)^n = \underbrace{\min\{x, y\} + \min\{x, y\} + \cdots + \min\{x, y\}}_{n \text{ times}} = n \min\{x, y\} = \min\{nx, ny\} = x^n \oplus y^n.$$

Part Three: Polynomials

A colloquial definition that we might give our college algebra students is that a polynomial is "powers of variables, times coefficients, all added up." Namely, polynomials are built out of sums and products of monomials, and we have access to "sums" and "products" in the tropical setting. Given a collection of variables, say, $\{x, y, z\}$, we can build a **tropical monomial** via tropical multiplication of these variables. For example:

$$y \otimes x \otimes z \otimes x \otimes z \otimes y \otimes z \otimes y = x^2y^3z^3.$$

Decoded, this is the linear function

$$2x + 3y + 3z$$

from $\mathbf{R}^3 \to \mathbf{R}$. If we allow our monomials to have negative integer exponents (though not non-integers, as what results probably shouldn't be called a tropical "polynomial" anymore), then we get the resulting theorem.

Theorem 1. *The tropical monomials $\mathbf{R}^n \to \mathbf{R}$ are in one-to-one correspondence with linear functions with integer coefficients.*

We can build generic polynomials out of monomials, and the same is true in the tropics. Define a **tropical polynomial** to be a finite **R**-linear tropical combination of tropical monomials. For example:

$$\begin{aligned} p(x, y, z) &= \pi x^2y^3z^3 \oplus 5x^2y \oplus \sqrt{2}y^{-3}z \\ &= \min\{\pi + 2x + 3y + 3z, 5 + 2x + y, \sqrt{2} - 3y + z\}. \end{aligned}$$

is a polynomial $p : \mathbf{R}^3 \to \mathbf{R}$. The tropical polynomials are in fact a minimum of a finite collection of linear functions, with integer coefficients except for possibly constant terms. Furthermore, tropical polynomials $p : \mathbf{R}^n \to \mathbf{R}$ satisfy the following three properties:

1. p is continuous,

2. p is piecewise-linear with finitely many pieces, and
3. p is concave; i.e.,

$$p\left(\frac{x+y}{2}\right) \geq \frac{p(x)+p(y)}{2}$$

for all $x, y \in \mathbf{R}^n$.

After a bit of work, we actually get the following theorem:

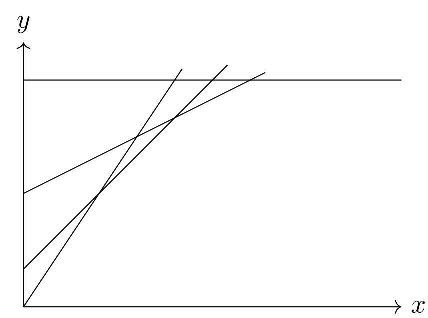
Theorem 2. *Every function that satisfies the above three properties is a minimum of a finite set of linear functions. Thus, tropical polynomials in n variables are in one-to-one correspondence with piecewise-linear concave functions on $\mathbf{R}^n$ with integer coefficients, except for possibly constant terms.*

Part Four: Geometry

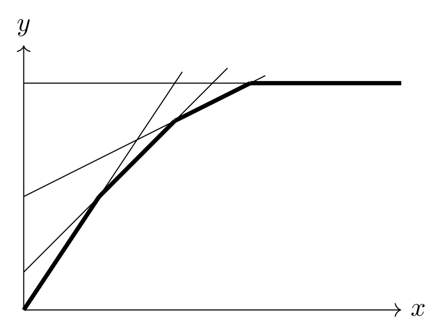
Now that we have polynomials, we want to graph them. Algebraic geometry is all about the graphs and zero sets of polynomials and the same is true in tropical geometry. Let's start with a basic example of a cubic polynomial in x. For instance:

$$\begin{aligned} p(x) &= x^3 \oplus 1x^2 \oplus 3x \oplus 6 \\ &= \min\{3x, 1+2x, 3+x, 6\}. \end{aligned}$$

Graphing $(x, p(x))$ in the xy-plane, we first graph the four linear polynomials.



Then the polynomial evaluated at a given x is the minimal y such that (x, y) is on one of the four lines.



Notice that p fails to be linear in finitely many points: $(1, 3)$, $(2, 5)$, and $(3, 6)$. Using this, we can factor p into three tropical linear factors

$$\begin{aligned} p(x) &= (x \oplus 1)(x \oplus 2)(x \oplus 3) \\ &= \min\{x, 1\} + \min\{x, 2\} + \min\{x, 3\}. \end{aligned}$$

In fact, in particular for a general cubic $ax^3 \oplus bx^2 \oplus cx \oplus d$, as long as $b - a \leq c - b \leq d - c$, we have effectively the same graph, rescaled, and the cubic factors into three linear terms

$$a(x \oplus (b-a))(x \oplus (c-b))(x \oplus (d-c)).$$

In general, we do have a "Tropical Fundamental Theorem of Algebra" of sorts: every one-variable tropical polynomial function can be uniquely factored as a tropical product of tropical linear functions. The word "function" is important, because two different tropical polynomials can represent the same function. For example, both $x^2 \oplus ax \oplus 2$ and $x^2 \oplus bx \oplus 2$ factor as $(x \oplus 1)(x \oplus 1)$, provided $a, b \geq 1$.

The situation gets even worse in two or more variables; irreducible factorization is no longer unique. For example:

$$(x \oplus 0)(y \oplus 0)(xy \oplus 0) = \min\{x, 0\} + \min\{y, 0\} + \min\{x + y, 0\}$$

is a tropical product of irreducibles, but so is

$$(xy \oplus x \oplus 0)(xy \oplus y \oplus 0) = \min\{x + y, x, 0\} + \min\{x + y, y, 0\}$$

and these are equivalent.

We can also focus on tropical planar curves $\mathbf{R}^2 \to \mathbf{R}$. We can graph them by graphing their "zero set," just like we do in algebraic geometry. But what a zero set actually is becomes a little more complicated. We'll work in the one-dimensional case first. We saw, for a cubic $ax^3 \oplus bx^2 \oplus cx \oplus d$ satisfying a sort of discriminant condition $b - a \leq c - b \leq d - c$, we get three linear factors $(x \oplus (b - a))$, $(x \oplus (c - b))$, and $(x \oplus (d - c))$, so the "zero set" should be $\{b - a, c - b, d - c\}$. These are the points where our cubic was not linear.

In general, we define a zero set of a tropical polynomial $p : \mathbf{R}^n \to \mathbf{R}$ to be the hypersurface

$$\mathcal{H}(p) \coloneqq \{x \in \mathbf{R}^n \mid p \text{ is not linear at } x\}.$$

Since p is a minimum of finitely many linear functions, p is not linear at x if and only if x is a singular point; i.e., at x, p achieves its minimum at least twice.

A generic curve is of the form

$$C(x, y) = \bigoplus_{(i,j) \in \mathbf{Z}^2} c_{ij} x^i y^j,$$

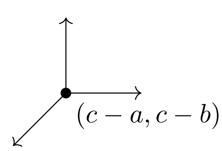
and the hypersurface $\mathcal{H}(C)$ is a finite graph embedded in $\mathbf{R}^2$. Furthermore, $\mathcal{H}(C)$ satisfies the following three properties:

1. $\mathcal{H}(C)$ has, in general, both bounded and unbounded edges,
2. the slope of every edge in $\mathcal{H}(C)$ is rational (taking infinite slope to be $1/0$), and
3. every vertex satisfies a "zero tension" condition, meaning if the slope of every edge out of the vertex is taken in lowest terms $\{b_1/a_1, b_2/a_2, \ldots, b_n/a_n\}$, then the sum of the vectors (a_i, b_i) is zero.

Let's see an example. Consider a tropical line $L(x, y) = ax \oplus by \oplus c$ with $a, b, c \in \mathbf{R}$. By definition,

$$\begin{aligned} \mathcal{H}(L) &= \{(x, y) \in \mathbf{R}^2 \mid L(x, y) = ax \oplus by \oplus c = \min\{a + x, b + y, c\} \text{ is not linear at } (x, y)\} \\ &= \{(x, y) \in \mathbf{R}^2 \mid \min\{a + x, b + y, c\} \text{ is satisfied by at least two of the terms}\}. \end{aligned}$$

This is three half-rays whose source is $(c - a, c - b)$ emitting north, east, and southwest.



We achieve zero tension since $(0, 1) + (1, 0) + (-1, -1) = (0, 0)$. We can check this with an explicit example: the line $L(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$ could have two or more terms achieving the minimum if:

1. $x = y \leq 1$,
2. $x \geq 1$ and $y = 1$, or
3. $y \geq 1$ and $x = 1$.

This gives us a center of $(1, 1)$ and our three rays: southwest, east, and north, respectively.

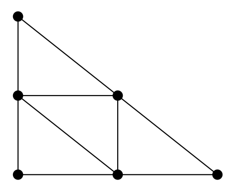
We have the following algorithm for drawing $\mathcal{H}(C)$ for any curve C.

1. Take every term $c_{ij}x^iy^j$ appearing in C.
2. Plot the point $(c_{ij}, i, j) \in \mathbf{R}^3$.
3. Take the convex hull of all such points.
4. Project the convex hull to $\mathbf{R}^2$ via $(x, y, z) \mapsto (y, z)$. The resulting projection is a planar convex polygon which has been subdivided into smaller polygons.
5. Take the negative of this projection; i.e., flip over the x- and y-axes.
6. $\mathcal{H}(C)$ is the dual graph to the flipped subdivision.

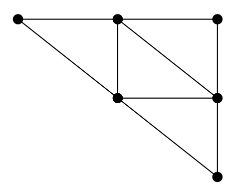
It'd be a bit overkill, but applying this process to a line $L(x, y) = ax \oplus by \oplus c$, we see that we have

1. $\{ax, by, c\}$
2. $\{(a, 1, 0), (b, 0, 1), (c, 0, 0)\}$
3. The convex hull of these three points is the hollow triangle determined by those three points.
4. The projection is a triangle, with trivial subdivision.
5. We flip this triangle over the x- and y-axes.
6. The dual graph to this triangle is a single vertex with three emanating rays.

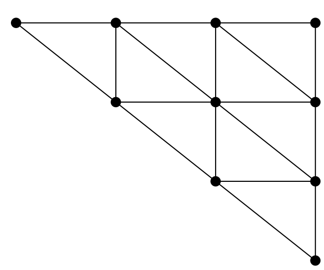
Exercise 3. Show that a general quadratic $Q(x, y) = ax^2 \oplus bxy \oplus cy^2 \oplus dx \oplus ey \oplus f$ with "discriminant condition" $2b \leq a + c$, $2d \leq a + f$, and $2e \leq c + f$ has planar convex polygon projection



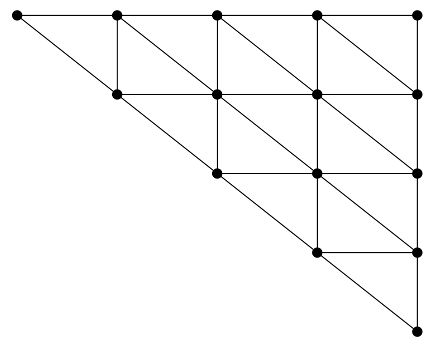
and thus $\mathcal{H}(Q)$ is dual to the following.



I believe, up to an appropriate discriminant condition, that a polynomial p of order n has $\mathcal{H}(p)$ dual to a triangle with $n + 1$ vertices on each side completely subdivided into smaller triangles. That is, if $n = 3$ and a discriminant condition is satisfied, then $\mathcal{H}(p)$ is dual to

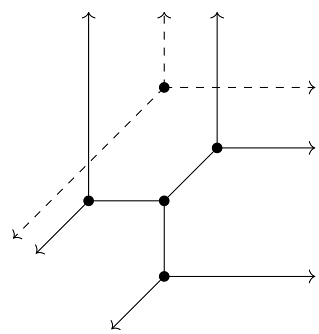


If $n = 4$ plus a discriminant condition, then $\mathcal{H}(p)$ is dual to



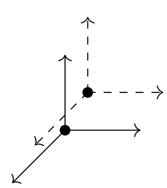
Et cetera.

We can solve tropical polynomial equations much in the same way that we solve polynomial equations. In fact, notice what happens when we ask for the intersection between a generic quadratic as above and a line. We get a picture like, for instance:



There are two intersection points. In fact, no matter where how you intersect a tropical quadratic and a tropical line, as long as they meet transversely, you're going to have two intersection points.

More simply, two lines intersect in one point:



Indeed, two quadratics intersect at four points, and in fact:

Theorem 4 (Tropical Bézout)**.** *Two tropical polynomials of degree d_1 and d_2 intersect transversely at $d_1 \cdot d_2$ points.*

Consequently, two general points determine a tropical line, five general points determine a tropical quadratic, etc. We can draw an example picture fairly easily. Fix one point and see how we can determine a line via one other point in general position.

1. If the left point is higher than the right point, draw lines straight down from the left point and straight left from the right point until they meet; that's your vertex.
2. If the right point is higher than the left point, draw lines straight diagonal up from the left point and straight left from the right point until they meet; that's your vertex.
3. If the points are on the same horizontal, then the points aren't general.

Wildly enough, even more algebraic geometry ports over to the land of tropical geometry. There is a tropical version of Riemann-Roch as well. There is a tropical degree-genus formula. You can even define tropical elliptic curves, and there is a group law on them!

In fact, you can take any algebraic variety X (which you're welcome to think of as just a solution set of regular polynomials) and produce from X a tropical variety, a solution set of $\mathcal{H}(p)$s. Sometimes questions in one setting are easily answered in the other, so having a way to translate like this is very nice. Furthermore, studying tropical geometry is, as we've seen, essentially studying graphs and combinatorics through an algebraic lens, meaning there are potentially whole hosts of questions which might be easily stated or answered just by changing the framing. Tropical geometry even has connections to non-archimedean fields, like the p-adics.